Q1. (8 marks)
Give examples of the following:

(a) A commutative subring of $M_2(\mathbb{R})$;

(b) A nilpotent polynomial of positive degree in $Z_{12}[X]$;

(c) A left ideal of a ring that is not a right ideal;

(d) Two nilpotent elements whose sum is not nilpotent.

Solution:

(a) $\left\{ \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix} : r \in \mathbb{R} \right\}$;

(b) $6x$

(c) $\left\{ \begin{bmatrix} a_{11} & 0 \\ a_{21} & 0 \end{bmatrix} : r \in \mathbb{R} \right\}$;

(d) Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Then $A^2 = B^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ so that A, B are nilpotent and $A + B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. We can calculate that $(A+B)^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

□

Q 2. (5+7 marks) Let A be a commutative ring with unity.

(i) Prove that an ideal M is maximal if A/M is a field.

(ii) If P is a prime ideal such that A/P is a finite ring, then prove that P must be maximal.

Solution:

(a) Note that A/M is a commutative ring with unity, where the addition and multiplication on A/M are given by $a + M + b + M = (a + b) + M$ and $a + M + b + M = ab + M$ for all $a, b \in A$. Now, let $a \in A \setminus M$. It is sufficient to show that $a + M$ has an inverse. Now consider the ideal $\langle a, M \rangle$ generated by a and M. Then $\langle a, M \rangle = \{m + ab : m \in M, b \in R\}$. Since $a \in \langle A, m \rangle \setminus M$, by maximality ideal property of M, we can ensure that $1 \in \langle A, m \rangle$. Thus $1 = ab + m$ for some $b \in A, m \in M$. This implies that inverse of $a + M$ is $b + M$.

(b) To show P is a maximal ideal in A, it is sufficient to show that A/P is field. Let $a + P, b + P \in A/P$ such that $ab + P = P$. Then $ab \in P$. Since P is a prime ideal, either $a \in P$ or $b \in M$. In other words, either $a + P = P$ or $b + P = P$. Thus A/P is a domain. Now we prove that A/P is field. Let $a + P$ be a non zero element of A/P. Since A/P is domain and A/P is finite, there exists $n > 0$ such that $a^n + P = 1 + P$. This implies that $a^{n-1} + P$ is the inverse of $a + P$. Hence A/P is a field. □

Q 3. (5+5 marks)

(i) Determine, with proof, all the idempotents of the ring $R = C([0,1], \mathbf{R})$ of continuous real-valued functions on $[0,1]$.

(ii) Let A be a commutative ring with unity. If $f = a_0 + a_1X + \cdots + a_nX_n \in A[X]$ is a unit, prove that a_n is nilpotent in A.

Solution:

(i) Let $f \in C([0,1], \mathbf{R})$ be an idempotent. Since idempotents of $\mathbf{R}$ is either 0 or 1, thus the range of f is the set $\{0,1\}$. Since f is continuous and $[0,1]$ is connected set, thus range of f has to be connected. Which implies that either $f = 1$ or $f = 0$.

(ii)

□

Q4. (4+6 marks)

(i) Let R be the ring

$$\mathbf{Z}[i;j;k] = \{a + bi + cj + dk : a;b;c;d \in \mathbf{Z}\}$$

of integral quaternions. Find its group of units.

(ii) Find all square roots of -1 in the ring

$$\mathbf{H} = \{a + bi + cj + dk : a;b;c;d \in \mathbf{R}\}$$

of real quaternions.

Solution: Note that 1 is the unit of $\mathbf{Z}[i;j;k]$.

(i) Let $q_1 = a_1 + b_1i + c_1j + d_1k$ be an unit element of $\mathbf{Z}[i;j;k]$. Then there exists $q_2 = a_2 + b_2i + c_2j + d_2k \in \mathbf{Z}[i;j;k]$ such that $q_1q_2 = 1$. Now

$$\begin{aligned} q_1q_2 = (a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2) + &(a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2)i + \\ &(a_1c_2 + c_1a_2 - b_1d_2 + d_1b_2)j + (a_1d_1 + d_1a_2 + b_1c_2 - c_1b_2)k. \end{aligned} \tag{1}$$

Thus $q_1q_2 = 1$, implies that

$$\begin{aligned} a_1a_2 - b_1b_2 - c_1c_2 - d_1d_2 &= -1 \\ a_1b_2 + b_1a_2 + c_1d_2 - d_1c_2 &= 0 \\ a_1c_2 - b_1d_2 + c_1a_2 + d_1b_2 &= 0 \\ a_1d_1 + b_1c_2 - c_1b_2 + d_1a_2 &= 0. \end{aligned} \tag{2}$$

Since all solutions of Equation (2) has to be integer, it is not vary hard to show that all the units of $\mathbf{Z}[i;j;k]$ are the set $\{\pm 1, \pm i, \pm j, \pm k\}$. Hence all the units of $\mathbf{Z}[i;j;k]$ form a group which is isomorphic to $\mathbf{Q_8}$.

(ii) Let $q = a + bi + cj + dk$ be an unit of $\mathbf{Z}[i;j;k]$ such that $q^2 = -1$. It follows that

$$\begin{aligned} a^2 - b^2 - c^2 - d^2 &= -1 \\ ab &= 0 \\ ac &= 0 \\ ad &= 0. \end{aligned} \tag{3}$$

Now either $a = 0$ or b, c, d are zero. Since a is real, $a = 0$, and $b^2 + c^2 + d^2 = 1$. Since b, c, d are integers, the square root of -1 is the set $\{\pm i, \pm j, \pm k\}$. □

Q5. (12 marks)

(i) Find all units of the ring $\mathbf{Z}[\sqrt{-d}]$, where $d > 2$ is an integer.

(ii) Prove that the polynomial $X^{50} - 101101X13 + 110$ cannot take either of the values 33 and -33 for an integer value of X.

Solution: (i) Let $a + \sqrt{-d}b \in \mathbf{Z}[\sqrt{-d}]$. Now $a + \sqrt{-d}b$ unit implies $a^2 + db^2 = 1$. Since $a, b \in \mathbf{Z}$, and $d > 2$, thus $a = \pm 1$ and $b = 0$. □

Q6. (i) Consider the ring homomorphism $\varphi : C[X, Y] \to C[Z]$ defined by $X \mapsto Z^2$ and $Y \mapsto Z^3$. Prove that the kernel of $\varphi : C[X, Y] \to C[Z]$ is the principle ideal generated by $X^3 - Y^2$.

OR

(ii) Consider a ring homomorphism T from $\mathbf{R}$ to itself. Show that if T is not the zero map, T is identity on $\mathbf{Q}$ and that $T(x) \geq T(y)$ if $x \geq y$. Deduce that T is continuous.

Solution: [i] Let $I = \langle X^3 - Y^2 \rangle$. It is clear that $\langle X^3 - Y^2 \rangle \subseteq \ker(\varphi)$. Let $f(x, y) \in \ker(\varphi)$. Now $f(x, y)$ can be expressed as

$$f(x, y) = \Sigma_{k=0}^{n} f_{3k}(Y)X^{3k} + \Sigma_{k=0}^{n} f_{3k+1}(Y)X^{3k+1} + \Sigma_{k=0}^{n} f_{3k+2}(Y)X^{3k+2},$$

where f_i are the polynomials over Y. Since $X^3 + I = Y^2 + I$, thus $f(x, y) + I = \Sigma_{k=0}^{n} f_{3k}(Y)Y^{2k} + \Sigma_{k=0}^{n} f_{3k+1}(Y)Y^{2k}X + \Sigma_{k=0}^{n} f_{3k+2}(Y)Y^{2k}X^2 + I$. It follows that

$$f(x, y) = \Sigma_{k=0}^{n} f_{3k}(Y)Y^{2k} + \Sigma_{k=0}^{n} f_{3k+1}(Y)Y^{2k}X + \Sigma_{k=0}^{n} f_{3k+2}(Y)Y^{2k}X^2 + h(X^3 - Y^2),$$

where $h(X^3 - Y^2)$ is a polynomial over $X^3 - Y^2$ such that $h(0) \neq= 0$. Since $f(x, y) \in \ker(\varphi)$, thus

$$\Sigma_{k=0}^{n} f_{3k}(Z^3) + \Sigma_{k=0}^{n} f_{3k+1}(Z^3)Z^2 + \Sigma_{k=0}^{n} (f_{3k+2}(Z^3))Z^3.Z = 0.$$

Applying fundamental theorem of algebra sincerely, we get that $\Sigma_{k=0}^{n} f_{3k}(Z) = 0$ and $\Sigma_{k=0}^{n} f_{3k+1}(Z) = 0 = \Sigma_{k=0}^{n} f_{3k+2}(Z^3)$. Hence $f(x, y) = h(X^3 - Y^2)$ so that $f(x, y) \in \ker(\varphi)$. □

(ii) Now $1.1 = 1$, and $T : \mathbf{R} \to \mathbf{R}$ is a ring hommorphism, thus $T(1).T(1) = T(1)$. Since T is non zero $T(1) \neq= 0$ so that $T(1) = 1$. Now $T(1 - 1) = 0$, implies that $T(-1) = 1$. By induction hypothesis, $T(n) = n$ for all $n \in \mathbf{Z}$. Let $p, q \in \mathbf{Z}$ such that $q =\neq 0$. Then $qT(\frac{p}{q}) = T(q)T(\frac{p}{q}) = T(p) = p$. Therefore $T(\frac{p}{q}) = \frac{p}{q}$, and T is identity on $\mathbf{Q}$. Now let $x \leq y$. Then $Ty - Tx = (T((y - x)^{\frac{1}{2}}))^2$. Therefore $T(x) \leq T(y)$ whenever $x \leq y$.

To show continuity, let $\epsilon > 0$. Then there exists $q \in \mathbf{Q}$ such that $0 < q < \epsilon$. Now for $|x - y| < q$, we have $\pm T(x - y) \leq T(|x - y|) \leq T(q) < \epsilon$. Hence T is continuos. □